Percolation framework in Ising-spin relaxation

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We introduce a framework based on the percolation idea to investigate the relaxation under zero-temperature Glauber and outflow dynamics on $L \times L$ square and triangular lattices. This helps us to understand the appearance of a double time regime in the survival probability. We show that the first, short-time, regime corresponds to relaxation through droplets and the second, long-time, regime corresponds to relaxation through stripes. For both dynamics the probability that the system becomes ordered through droplets (which indicates fast relaxation) is about 2/3.

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Systems quenched from a disordered into an ordered phase (such as the Ising model quenched from initial temperature $T_0=\infty$ to final $T_F=0$) in the thermodynamic limit never reach the final ferromagnetic steady state. This is one of the reasons why the theory of phase ordering kinetics has remained a challenge for more than four decades (for a review, read [1]). Moreover, Spirin *et al.* [2] showed that even a simple two-dimensional Ising ferromagnet has a large number of metastable states with respect to zero-temperature Glauber dynamics [3] and, therefore, at zero temperature the system could get stuck forever in one of the metastable states that consists of alternating vertical or horizontal stripes—from now on we call it the stripe configuration (S). This is understood on the basis of the definition of zero-temperature Glauber dynamics, which involves picking a spin at random and flipping it according to the direction of a majority of its nearest neighbors. If there is no majority, the spin is flipped with probability 1/2. Thus a straight interface does not evolve. A slight difference between square and triangular lattices in the probability $P_{\text{str}}(\infty)$ that the system eventually reaches a stripe state was found in [4]: $P_{\text{str}}(\infty)=0.315$ and 0.344 on the square and triangular lattices, respectively. Moreover, in the case of the square lattice in about 0.04 of all simulations a diagonal stripe (DS) configuration appears [2].

Very interesting behavior is exhibited by the survival probability $S(t)$ that the system has not yet reached its final state by time t. On a semilogarithmic plot $S(t)$ lies on a straight line with a large negative slope and then crosses over to another line with smaller negative slope [2]. Recently, similar behavior of $S(t)$ was observed for Ising spins under outflow dynamics [5], which originally was introduced to describe opinion change in a society [6].

A number of social experiments have shown that, when faced with a strong group consensus, people often conform even if they believe that the group may be in error. However, even a single visible dissenter from the group's position emboldens others to resist conformity [7]. This observation was recently expressed in a simple one-dimensional "united we stand, divided we fall" model of opinion formation [6]. The model was later renamed the Sznajd model by Stauffer *et al.* [8] and generalized to a two-dimensional square lattice. In its two-dimensional version the model has found a number of social applications (for reviews, see [9–13]), but in this paper we investigate it from the theoretical point of view. The crucial difference between the Sznajd model and zero-temperature Glauber dynamics [3] is that information flows outward from the center nodes to the surrounding neighborhood and not the other way around—hence the name outflow dynamics. It should be mentioned that, although one-dimensional outflow dynamics obeys detailed balance, no finite-temperature version of the outflow rule has been proposed up till now. It seems that the temperature cannot be introduced into our dynamics without breaking the detailed balance condition, but further studies concerning this issue are definitely needed. Moreover, in contrast to Glauber dynamics, generalization of the one-dimensional rule to higher dimensions is neither straightforward nor unambiguous. Several types of two-dimensional outflow dynamics have been already introduced [5,8,9], and recently three of them have been investigated from the theoretical point of view [5]. For all three investigated outflow dynamics, a short- and a long-time regime have been observed. The short-time regime (fast relaxation) was observed for about 2/3 of all trials [5].

In this paper, we introduce a framework based on the percolation idea to investigate the evolution of the configuration under zero-temperature Glauber and outflow dynamics on two-dimensional square and triangular lattices (suggestions that percolation phenomena can influence zero-temperature dynamics have appeared already in [14]). This helps us to understand the appearance of two time regimes in the survival probability $S(t)$. We focus here only on one type of outflow dynamics defined below, but the same results could be obtained for other types of two-dimensional outflow dynamics investigated in [5]. Let us begin with the definition of the dynamics. The system consists of $L \times L$ Ising spins $S_i = \pm 1$ $(i=1,\dots,L^2)$ placed on a two-dimensional lattice with periodic boundary conditions. In the case of the square lattice, in each update a 2×2 panel of four neighbors is selected randomly. If all four spins in a panel are parallel then the panel flips its eight nearest neighbors to the unanimous direction of the four spins in the panel. In other cases, these eight neighbors are left unchanged. Similarly we define the dynamics on a triangular lattice (for details see [5]). Under outflow dynamics the system eventually always reaches a ferromagnetic steady state, in contrast to zero-temperature Glauber dynamics. For this reason outflow dynamics is sim-

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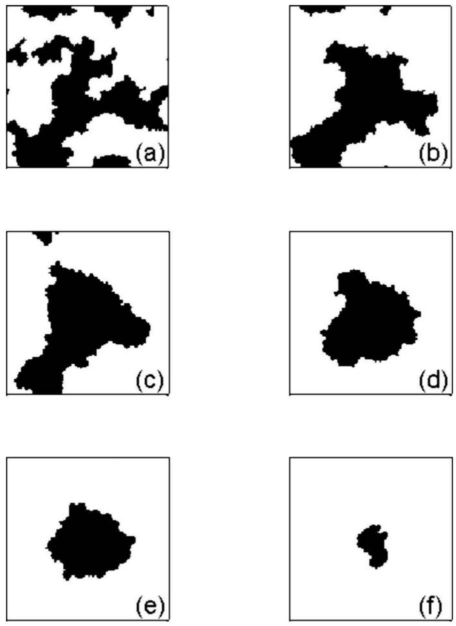

FIG. 1. Snapshots of the sample relaxation under outflow dynamics on a two-dimensional 100×100 square lattice from a random initial state consisting of 50% up spins after (a) 100, (b) 300, (c) 400, (d) 1000, (e) 1500, and (f) 2300 Monte Carlo steps (MCS). In this trial, after a relatively short time (about 300 MCS) a simply connected cluster (droplet) is formed.

pler to analyze and the percolation framework is easier to understand.

Let us begin with presenting two sample relaxations under our outflow dynamics on a 100×100 square lattice (see Figs. 1 and 2). Initially, the system consists of randomly distributed equal numbers of up (50%) and down (50%) spins. After a relatively short time in each relaxation only one of two types of configurations is created—droplets (Fig. 1) or stripes (Fig. 2). In the stripe configuration one of the stripes eventually breaks at one point to form a droplet and from this moment the evolution of the system leads very quickly to the ferromagnetic steady state. This observation led us to the following postulate: A system quenched from a disordered to an ordered phase evolves through droplets (fast relaxation) or stripes (slow relaxation). The first, short-time, regime in the survival probability $S(t)$ corresponds to relaxation through droplets, and the second, long-time, regime to relaxation through stripes. We expect that the above postulate is valid not only in the case of outflow but also zero-temperature Glauber dynamics. To confirm this postulate we introduce now a framework based on the percolation idea.

In the following the quantity of central interest will be the connectivity of clusters of given spins (up or down) in a specified direction (top to bottom or left to right). We say that the connectivity is nonzero (1) in a given direction (e.g., left-right) if two opposite edges of the system (left and right) can be connected via a continuous path composed of the given spins [e.g., for spins up we denote the left-right connectivity as $P_{LR}(\uparrow)=1$ and so on]. For one type of spins there are four distinct possibilities of overall connectivity: zero in both directions (00), nonzero in one direction (01 or 10), and nonzero in both (11). As we deal with two types of spins,

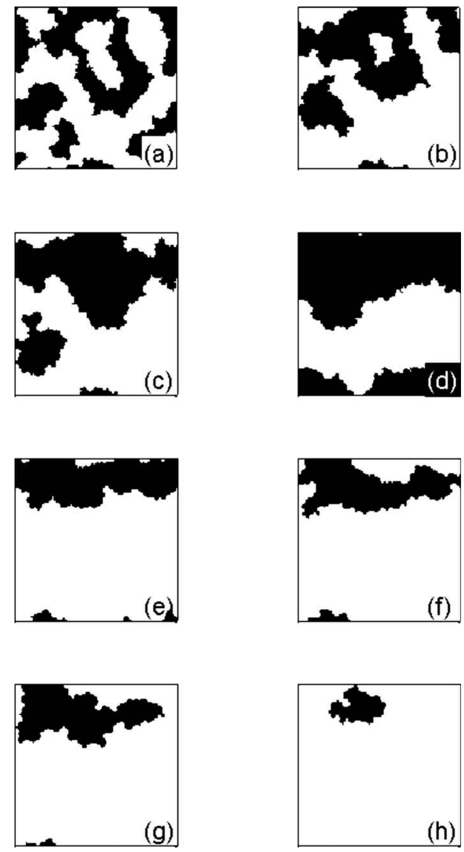

FIG. 2. Snapshots of the sample relaxation under outflow dynamics on a two-dimensional 100×100 square lattice from a random initial state consisting of 50% up spins after (a) 100, (b) 300, (c) 500, (d) 5000, (e) 14 000, (f) 15 000, (g) 15 100, and (h) 16 000 Monte Carlo steps (MCS). In this trial after a relatively short time (about 1000 MCS) the stripe configuration is formed. Eventually, one of the stripes breaks at one point to form a simply connected cluster, and from this moment the evolution of the system leads very quickly to the final state with all spins in the same state.

there are (at least in principle) 16 various combinations of connectivity possible. In the hard wall boundary conditions some configurations are forbidden, e.g., up spins connected vertically while down spins are connected horizontally. With periodic boundary conditions, however, all possibilities are valid; see Fig. 3 for a short review. Some configurations [the first four—the chessboard, stripes (horizontal or vertical) on chessboard, and odd configurations] are so exceptional that we have never observed them in real simulations. The main idea of the percolation framework analysis of system dynamics consists in counting how much time the system spends in each configuration in its history from the random initial state toward the steady final state. In order to obtain information in as clear and compact way as possible, for each simulation sample, we provide four cumulative times spent by the system in the following configurations: droplet (D), stripes (S), diagonal stripes (DS), and transient (T). The diagonal stripes configuration is generally defined as having full connectivity in both directions (horizontal and vertical) for both spin orientations (up and down): [11-11]—see Fig. 3. Its name comes from the simplest example of this configuration in the

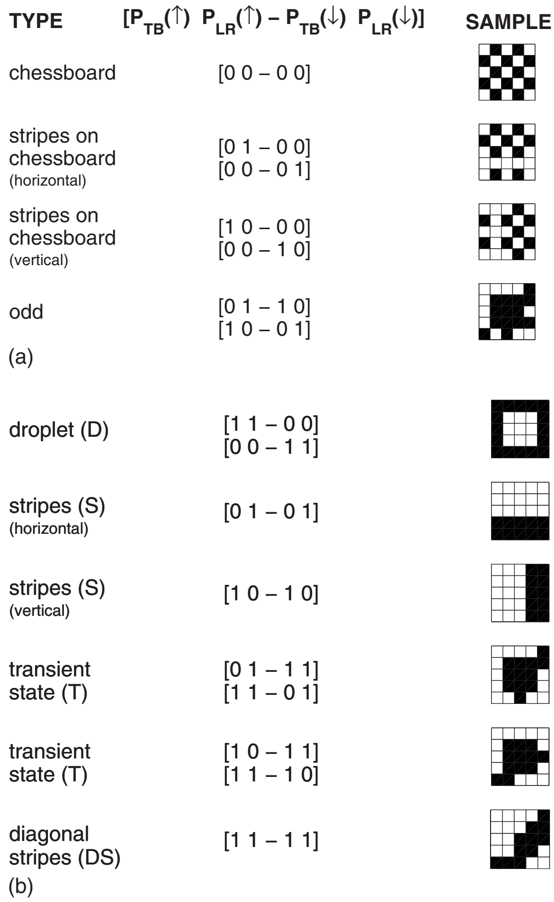

FIG. 3. All possible configurations with respect to connectivity of up and down spins on a lattice with periodic boundary conditions. In the middle column a digit 1 appearing at a given position indicates the connectivity of spins up or down in the vertical (TB) or horizontal (LR) direction. The first four types, although theoretically possible, do not appear in real simulations.

shape of alternating stripes angled at 45° to the horizontal. Here the periodicity of the boundary conditions is crucial, otherwise there is no possibility of connectivity in both directions for both spin components. The last configuration's name (T) comes from the fact that these states do not last long and are possibly a by-product of a transition between more stable configurations. In order to speed up the simulations, we decided to make a check of the configuration type not continuously, but at certain times. We verified that our choice of checking time interval (=1 MCS) did not affect the quality of the results.

Application of the percolation framework analysis to our outflow dynamics helps in understanding the shape of the survival probability obtained in previous work [5]. The data confirm our postulate of either fast evolution through droplets or slow evolution through stripes. The times spent by the system in various configurations are presented in Fig. 4 for our outflow dynamics on a periodic square lattice of size $L=100$. There are shown data collected from $N=1000$ simulations. For each simulation the relaxation time is the abscissa of the symbols. For each configuration type appearing (D, S, DS, T) its cumulative time is the ordinate. Thus for each simulation there are four points at the same abscissa value,

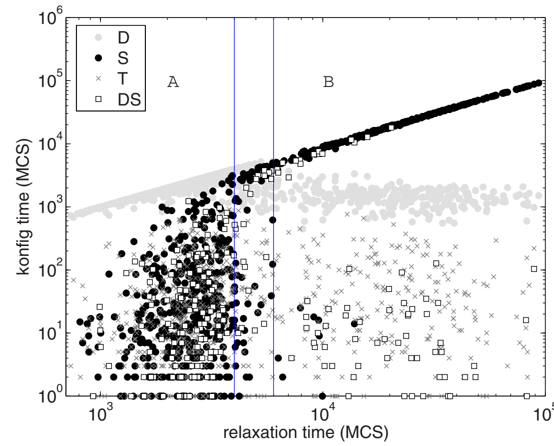

FIG. 4. (Color online) Relaxation under outflow dynamics on two-dimensional 100×100 square lattice from random initial state consisting of 50% up spins; the data from $N=10^3$ simulation are presented. Symbols show how much time the system spends in given configuration type before reaching the final state. For short relaxation times ($\tau<4\times10^3$ MCS) the evolution goes mostly through the droplet state (D), while for longer relaxation times nearly 100% of the time is spent in the stripe configuration (S) [occasionally in the diagonal stripe (DS) configuration]. The crossover time (here approximately 4×10^3 MCS) coincides with the time where the change of slope appears in the survival probability [5].

representing the contributions of particular configuration types to the total relaxation time. For example, let us consider a simulation having relaxation time 10 000 MCS. Let us assume that during the evolution toward its final state the system spent 2000 MCS in D configurations, 7950 MCS in S configurations, 45 MCS in T configurations, and 5 MCS in DS configurations. Thus resulting from this particular simulation there appear four points on the plot having the following coordinates: (10 000, 2000), (10 000, 7950), (10 000, 45), and (10 000, 5). The proximity of a symbol to the line $y=x$ indicates that the system dwells in the corresponding configuration most of the time until relaxation. The log-log setting of the plot makes it possible to bring out more details interesting for further analysis. Let us assume that a particular configuration type (say X) dominated the system history until the final state in all simulations with relaxation times from some interval. Thus one would see that symbols corresponding to this configuration type X would group high in the plot along the line $y=x$ (or very close to this line) on the mentioned interval. The other, much rarer configuration types would be found as symbols at the bottom of the same plot. On the other hand if there was a case of equally long-lasting configuration types (say, each types D, S, T, and DS took 25% of the relaxation time), the points would all lie well below the line $y=x$ (this is not the case in the considered set of data, however). There is yet another possibility—in different simulations of given relaxation times various configuration types dominate. In such a case it could be seen on the plot that the different symbols approach the line $y=x$ (in our case we have there a transition region; see further in the text).

All simulations considered in Fig. 4 naturally split into two sets—one ($\mathcal{A}$), for which the droplet configuration dominates (this is the case for all simulations with relaxation time smaller than 4×10^3 MCS) and the second ($\mathcal{B}$), where the system spends most of the time in the stripe configuration (here belong all simulations with relaxation times greater than 6×10^3 MCS). To the latter also belong the rare simulations for which long-lived diagonal stripes are observed. There is also a third, transition region (relaxation times between 4×10^3 and 6×10^3 MCS) consisting of simulations for which the dominant configuration type is not unique. Then there is a considerable probability of finding simulations with various dominant configuration types.

In the set $\mathcal{A}$ (short relaxation times) we attribute different values of the droplet dwelling time to different sizes of the droplet arising from the random initial state (for bigger droplets the relaxation time is longer [5]). The dynamics of the samples from the set $\mathcal{B}$ is different: most of the time the system spends in the stripe configuration, after which the stripe breaks and the resulting droplet evolves according to the previous scenario (pertaining to the set $\mathcal{A}$). In this case the droplet part of the total time remains at the same level (about 1.5×10^3 MCS on Fig. 4); this is because the droplet arising from breaking the stripe has more or less the same size (of order of half the size of the system). In the case of the stripes their dwelling time has a much broader distribution, resulting not only from the differences in width of the stripes that arise from the random initial state, but mainly from behavior similar to a Brownian random walk. For the stripe configuration the rather straight interface between clusters of spins with different orientations has equal chance to move in either direction (for the droplet the direction of the interface movement is always toward its center). The characteristic time limiting from above the set $\mathcal{A}$ (here about 4×10^3 MCS) coincides exactly with the time of change in the slope of the survival probability [5]. These two regimes of exponential dependence correspond to evolution through either the droplet or stripe configuration (the former are interestingly always about 2/3 of all cases).

From the above analysis there appears the following scenario for the dynamics of the system. At the first stage, when the system starts its evolution from a totally random state with 50% spins up and 50% spins down (i.e., quenched from infinite temperature) small clusters tend to either grow or disappear and the characteristic length in the system (the mean width of the clusters) approaches the system size. The interface between clusters of opposite spins gets smoother and smoother. At a certain (rather short) time the state of the system belongs to either the droplet, stripe, or diagonal stripe configuration. In the first case (D) it is known [5] that the droplet relaxes to the final steady state relatively fast via shrinking (it has been proved already that every smooth closed curve in the plane asymptotically approaches a shrinking circular shape [15,16]). In the case when the system in the first stage is in the stripe configuration, the evolution is much slower (stripes at some points get thicker, at others get thinner). One of the stripes eventually narrows to make a break, the cluster becomes simply connected, and the configuration switches to a droplet. The only configuration not discussed yet—the transient one (T)—appears for short periods and only either at the beginning of the simulations

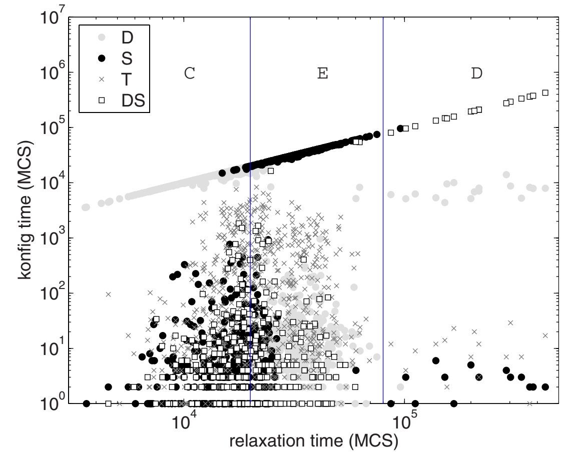

FIG. 5. (Color online) Relaxation under Glauber (inflow) dynamics on two-dimensional 100×100 square lattice from a random initial state consisting of 50% up spins; the data from $N=10^3$ simulation are presented. Symbols show how much time the system spends in a given configuration type before reaching the final state. For short relaxation times ($\tau<2\times10^4$ MCS) the evolution goes mostly through the droplet state (D). For much longer relaxation times ($>8\times10^4$ MCS) nearly 100% of the time is spent in the diagonal stripe configuration (DS). For intermediate relaxation times the stripe configuration (S) dominates. The crossover time (here approximately 2×10^4 MCS) coincides with the time where the change of slope appears in the survival probability [5].

(when the system "decides" whether to go through the stripe configuration or directly through the droplet configuration) or at switching times, when the system changes its configuration (e.g., DS$\rightarrow D$). Our extensive simulations proved that all above statements remain valid for outflow dynamics on a triangular lattice as well.

In the case of Glauber dynamics the overall dynamics characteristic is somewhat similar, but a bit more complicated. This is because in this dynamics the regular stripe configuration (with straight line interfaces) is the final one (in contrast to the outflow dynamics, where it always decays to the ferromagnetic state with all spins parallel). In Fig. 5 there are presented data for Glauber dynamics simulations, but here the relaxation time is measured until the system reaches any of its final states (including regular stripes). There is a natural partition into three sets of simulations: set $\mathcal{C}$ with evolution mostly through droplets (it corresponds to the set $\mathcal{A}$ of the previous dynamics), set $\mathcal{D}$ with the evolution leading mostly through diagonal stripes (somewhat similar to the set $\mathcal{B}$), and set $\mathcal{E}$ of samples leading to the final regular stripe configuration, characterized by the absolute majority of stripe configurations.

Cases from the set $\mathcal{C}$ correspond exactly to the previously described set $\mathcal{A}$ of outflow dynamics. The only difference between the set $\mathcal{D}$ and the previously considered set $\mathcal{B}$ is that in the set for Glauber dynamics there is evolution only through diagonal stripes, since the horizontal and vertical stripes no longer decay to the ferromagnetic state and they form the new set $\mathcal{E}$. Depending whether the system decides at an early stage to evolve through the droplet configuration, stripes, or diagonal stripes, we got the shortest, moderate, and longest relaxation times, respectively. In the plot of survival probability of Ref. [2] the change of the slope coincides with the largest time in the set $\mathcal{C}$ (here 2×10^4 MCS). The relative size of the set $\mathcal{C}$ (the probability that the system chooses the droplet configuration) is here also $2/3$, as it was in the case of outflow dynamics. This universal constant for outflow dynamics [5] and for Glauber dynamics on a square lattice as well as on a triangular one (we checked that Glauber dynamics on a triangular lattice also conforms with the previous conclusions for a square lattice) must have some simple explanation, but unfortunately it needs further investigations. One can suppose that this property is of a fundamental nature for the broader class of zero-temperature dynamics considered in the literature [17,18].

We believe that the percolation framework we proposed in this paper could be used to study relaxation not only in the case of zero-temperature Ising-spin dynamics, but also in a much broader class of coarsening systems. Our method gives deeper insight into the relaxing system than the survival probability. It cannot describe configurations in detail, as was done, for example, in [19]. On the other hand, it gives general information on the system structure during relaxation, which may help to find some universal features of the dynamics investigated.

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